APPLICATION OF DUALITY THEORY IN THE ANALYSIS OF LINEAR PRODUCTION PLANNING PROBLEMS

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Abstract. *The paper presents the relevance of developing new methodological approaches for application of duality theory for more in-depth analysis of linear production planning problems. The theory of duality allows to find solutions to a dual problem in several ways, one of which is based on the construction of an inverse matrix. This matrix is obtained either from a simplex table involving the optimal plan or by algebraic methods from a matrix constructed on the basic vectors (corresponding variables are included in the basic of the optimal solution). In practice, finding an optimal solution to a linear programming problem by the iterative simplex method is a cumbersome process, the application of duality theory is possible only for problems in a small number of variables and constraints. In this paper we propose an algorithm that avoids routine calculations and finds an inverse matrix for further study of the production planning process. The algorithm is based on the reports of solving a linear programming problem built in MS Excel using the "Solver" add-in and the properties of the simplex method for solving linear programming problems. Studies have shown that this algorithm allows us to analyze the optimal production plan for linear problems in a large number of products, given production and demand constraints. The presented algorithm has a number of drawbacks, which in general do not significantly affect the process of finding the inverse matrix. Scientific research has shown that it is possible to further develop such methodological approaches to getting more complete information about the optimal solution of problems by the simplex method.*

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Introduction. Linear programming methods allow to obtain not only optimal values of variables. Any business entity is primarily interested in studying dynamic changes in economic systems, which may affect the development and functioning of companies. Changes in the conditions under which the model was built lead to a loss of its relevance. Companies face and overcome a constant set of challenges such as: changes in resource inventory, introduction of new technologies or equipment replacement, changes of pricing policy, a new product launch and/or termination of unprofitable type of products, etc. In these situations the optimal production plan will become either suboptimal or not applicable at all in a new environment. To study the impact of such economic situations on the optimal plan, experts use such mathematical tool as the theory of duality. It makes possible to determine the status of resources and the stability intervals of dual variables due to changes in scarce resources or the replacement of one scarce resource to another. The dual variables provide the opportunity to measure product profitability. Also, during mathematical modeling and sensitivity analysis, it will possible to identify changes in the parameters of the original model on the previously obtained optimal feasible solution in linear programming problem.

The development of new algorithms for solving the optimization problems based on the duality theory is considered in the paper (Auslender & Teboulle, 2003). The authors (Daskalakis, Deckelbaum & Tzamos, 2015) apply a duality-based framework for revenue maximization in a multiple-good monopoly. Elements of duality theory in game theory are used by the authors (Bagan, Calsamiglia, Bergou, & Phys, 2018).

In the paper (Gozlan, Roberto, Paul-Marie & Tetali,, 2017) the researchers prove a Kantorovich type duality theorem to minimize the overall transport costs. Khrushch L. (2018) provided economic and ecological production functions investigating the appropriate dual problems explicity to the deduced one-criterion problems. The group of researchers (Chernova, Titov, Chernov, Kolesnikova, Chernova & Gogunskii, 2019) focused attention on the generalization of the operations of conversion of the primal problem to the dual problem. In the paper (Vybrani pytannia kompiuternoho modeliuvannia protsesiv i yavyshch, 2022) the authors consider the application of mathematical modeling in different fields of science, social and economic relationship and the use of computer mathematics and digital applications to solve practice-oriented problems.

In the paper (Moroz, Zahorianskyi, Haikova & Kuziev, 2022) the researchers formulated a defect algorithm for a special transportation problem applying optimality conditions from the theory of duality in linear programming. In the paper (Kuzmychov, Chernetska & Shestakov, 2022) the researchers investigate two tools of sensitivity analysis for the solution of a transportation problem, determining the advantages of the SolverTable in comparison with the Solver add-in in MS Excel.

The scientists Blaga N. and Priyamak I. (2019) noted the application of the dual simplex method and elements of the duality theory to select an effective strategy for the development of an enterprise. Yushchenko N. (2021) conducts a study of the network to solve linear problems of minimizing the total cost of the project and applies elements of the duality theory to define dual variables as flows in a network with limited bandwidth.

Researchers (Martynova & Shevchenko, 2022). prove that any matrix game can be transformed to the dual linear programming problems, which allows to obtain an optimal solution to a direct problem for one player, and the solution of the dual problem will be the optimal strategy for the second player.

Stepanova (2019) uses elements of the duality theory to identify bottlenecks in the combined production flows of metallurgical production. The authors (Aloshyn, Panchenko & Prykhodko, 2021) show that the principle of duality can be used to reduce the time of analysis and calculation of an information-measuring system. The researchers (Zhurbenko & Chumakov, 2021) use the dual simplex method to solve a problem related to planning the modernization of the transport system.

Thus, having studied the papers of domestic and foreign scholars, we determine the relevance of further development of new algorithms based on the duality theory in order to use them for a deeper analysis of economic models. Simplex method is suitable for solving linear programming problem. It is possible to obtain the optimal solution using tabular form. However, in practice, specialists must take into account more than a dozen different types of products. Usually, the solving of a linear programming problem is carried out according to the following scheme (Fig. 1). The implementation of the algorithm shown in Fig. 1, even using formulas in MS Excel, is quite cumbersome, since the algorithm is iterative in nature and requires a large amount of calculations and computations. This paper proposes an algorithm developed by the authors that avoids the drawbacks of solving a linear programming problem by the simplex method, but provides an opportunity to conduct an economic and mathematical analysis of the impact of objectively determined estimates on the optimal plan using only the built-in "Solver" in MS Excel.

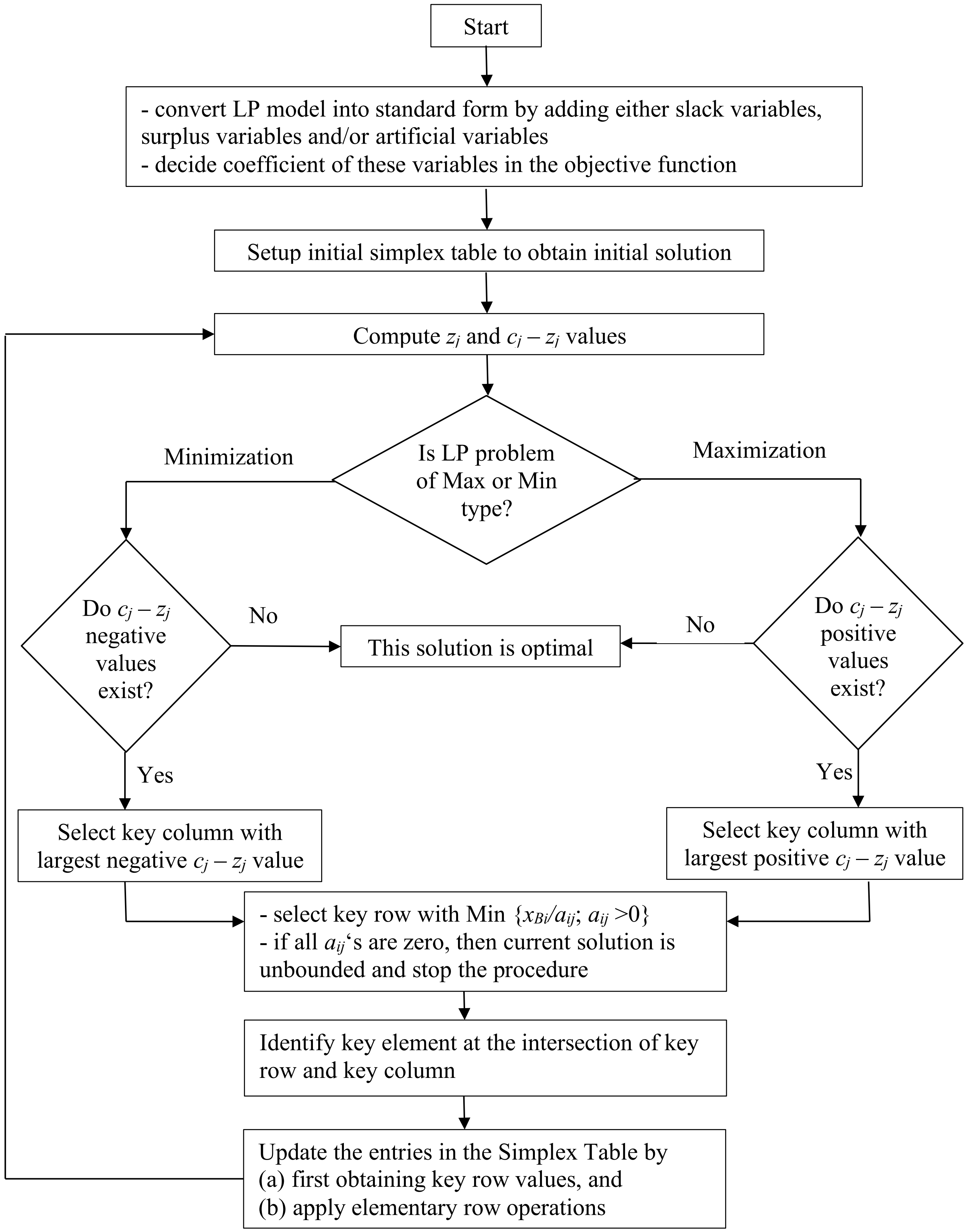

Fig. 1 Flow Chart of Simplex Algorithm

Source: (Sharma, 2017, p. 113)

The capabilities of the built-in functions of MS Excel spreadsheet application allow to conduct only an initial analysis of optimization problems and their results. Application of the duality theory with the capabilities of MS Excel allows to conduct an analysis that determines the impact of all hidden factors on the economic process under study, and as a result, to identify its negative aspects and develop ways to solve them.

Materials and Methods. Each linear programming problem can be compared to some other linear programming problem, called a dual. A dual problem is composed in accordance with the following rules:

- if the primal problem is a maximization problem, the dual to it is a minimization problem;
- the number of variables of the primal problem is equal to the number of constraint solutions of systems of inequities;
- the constraint system of the maximization problem has all inequalities of the type « ≤ », the minimization problem has inequalities of the type « ≥ »;
- matrix of the coefficients with unknowns in an inequality of both problems are transposed (the rows of one matrix correspond to the other columns);
- inequalities of primal problem are equal to the coefficients of the objective function of other problem. With a pair of symmetric dual problems, we have n + m related pairs of conditions.

If the constraint system of a problem contains an equality (one or more), then the dual problem to it also exists and is solved according to the same rules with one difference (Ladohubets & Finohenov, 2019).

In the duality theory, there are several ways to obtain solutions to a dual problem if there is a solution to a primal problem, one of them is: the vector of the optimal solution of the dual problem (Y*) is calculated due to the duality relation using first theorem as follows (Vitlinskyi, Tereshchenko. & Savina, 2016, p. 73):

$$Y* = \bar{C}_b D^{-1}, \qquad (1)$$

where $\bar{C}_b$ – a row vector, which consists of the coefficients of the objective function of the primal problem with the variables that are basic in the optimal solution;

D^{-1} – matrix inversed to matrix D, consists of the basis vectors of the optimal solution, the components of which are taken from the initial plan of the problem.

This inverse matrix D^{-1} allows us to analyze the impact of changes in resources on production. In other words, an economist does not need to implement a rather cumbersome iterative process of the simplex method in order to find a simplex table, containing an optimal solution of a linear production planning problem with a large amount of initial data and given resource constraints. In Fig. 2 shows a block scheme of the algorithm for obtaining the inverse matrix, which eliminates the cumbersome process of solving a linear programming problem by the simplex method. According to the flowchart above (Fig. 2), the researcher must perform the following steps:

1. Find the solution of a linear model in MS Excel using the Solver add-in and generate Answer report and Sensitivity report.
2. Identify the unprofitable types of products, in the optimal solution such types of products have zero values (Answer report). Identify inequalities in the model as a separate type of constraints on the admissibility of the solution for the unprofitable products and take them into account when building the canonical form.
3. Represent the canonical form of the linear programming problem.

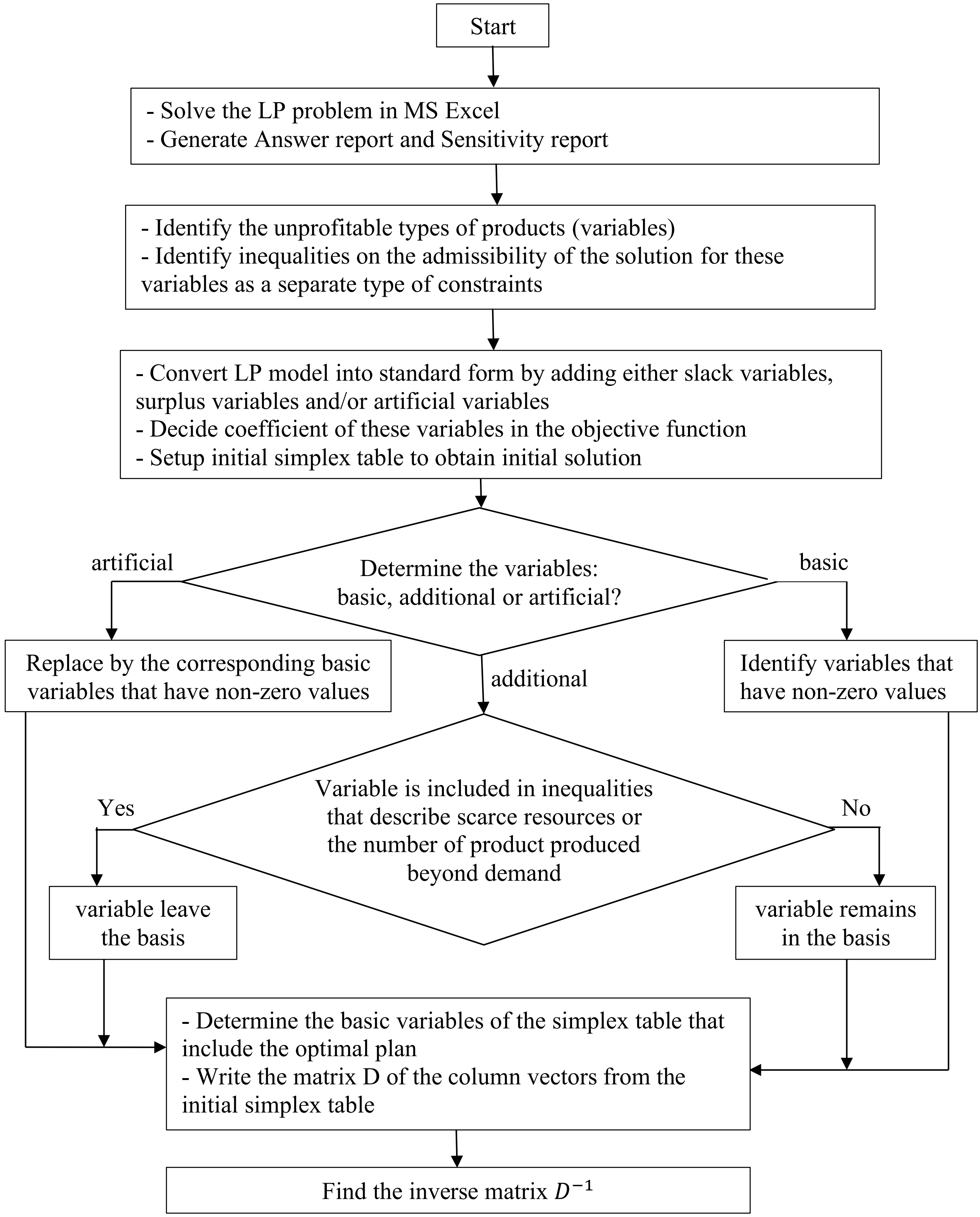

Fig. 2 Block scheme of the algorithm for finding the inverse matrix

Source: created by the authors

4. Write initial simplex table.

5. Determine the basic variables of the simplex table that include the optimal plan (the order doesn't matter if the model is linear):

5.1. Identify additional variables that correspond to scarce resources, unprofitable products in the constraints with artificial variables. Identify additional variables that correspond to the types of products that are produced at the upper bound on demand (the most profitable products for the enterprise). The identified variables will not be the basic in the simplex table containing the optimal solution and will be equal to zero.

5.2. The artificial variables are replaced by the corresponding basic variables of the problem that have non-zero values in the equations of the canonical form.

5.3. Identify additional variables that correspond to non-scarce resources and types of products whose production does not equal the upper bound on demands. The variables will remain basic.

5.4. The basic will also include the basic variables of the problem that have non-zero values and are not defined in the previous stages of the algorithm.

6. Write the matrix D of the column vectors from the initial simplex table in accordance with the system of basic variables.

7. Find the inverse matrix D^{-1}.

8. Check if an inverse matrix is correct by calculating the system of objectively determined estimates using formula (1) and comparing it with the shadow prices in the Sensitivity report.

9. Analyze the impact of objectively determined estimates on the value of objective function and the impact of simultaneous changes in multiple constraints.

The proposed algorithm can be applied only to non-degenerate dual linear programming problems that have a nonempty set of feasible solutions.

Results and Discussion. Authors implement the proposed algorithm to the different types of production planning regarding resource limitation and existing demand.

Suppose it is required to make a production plan of four types of products (P_1, P_2, P_3, P_4) regarding the existence of three types of raw materials (RM_1, RM_2, RM_3) in order to maximize the product revenue (Table 1).

Table 1

Data for a linear production planning problem

Types of resources	Raw materials usage per unit				Raw materials inventory, conventional units
	P_1	P_2	P_3	P_4	
RM_1	2	3	4	3	220
RM_2	3	2	2	2	250
RM_3	4	3	1	2	270
Product revenue, monetary units	17	16	15	12	

Source: created by the authors

Suppose that there is 11 units per pre-order for the fourth type of product. Construct the economic-mathematical model of this problem.

Firstly, introduce the notation:

x_i - the amount of products of the i-type, i = 1,2,3,4.

Then the objective function that expresses the maximization of revenue will be as follows:

$$Z = 17x_1 + 16x_2 + 15x_3 + 12x_4 \rightarrow \max$$

Subject to the constraints:

$$\begin{cases} 2x_1 + 3x_2 + 4x_3 + 3x_4 \leq 220 - raw\ material\ RM_1 \\ 3x_1 + 2x_2 + 2x_3 + 2x_4 \leq 250 - \ raw\ material\ RM_2 \\ 4x_1 + 3x_2 + x_3 + 2x_4 \leq 270 - raw\ material\ RM_3 \\ x_4 \geq 11 - production\ plan\ of\ the\ fourth\ type\ of\ product \end{cases}$$

$$x_i \geq 0,\ i = 1, 2, 3, 4.$$

Firstly, according to the algorithm, we find the solution of the linear model in MS Excel using the built-in "Solver" and generate Answer report (Fig. 3) and Sensitivity report (Fig. 4).

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$G$12	Revenue Total	0	1379,5

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$10	Production 1	0	57,5	Contin
$C$10	Production 2	0	0	Contin
$D$10	Production 3	0	18	Contin
$E$10	Production 4	0	11	Contin

Constraints

Cell	Name	Cell value	Formula	Status	Slack
$G$13	Raw material 1	220	$G$13<=$I$13	Binding	0
$G$14	Raw material 2	230,5	$G$14<=$I$14	Not Binding	19,5
$G$15	Raw material 3	270	$G$15<=$I$15	Binding	0
$B$10	Production 1	57,5	$B$10>=$B$11	Not Binding	57,5
$C$10	Production 2	0	$C$10>=$C$11	Binding	0
$D$10	Production 3	18	$D$10>=$D$11	Not Binding	18
$E$10	Production 4	11	$E$10>=$E$11	Binding	0

Fig. 3. Answer report

Source: compiled by author's calculations

Secondly, we determine from the data in Figs. 3 and 4, that according to the optimization result, the second type of product is unprofitable and its forced inclusion in the production plan will lead to a decrease in total revenue by 1,36 monetary units per unit. The fourth type of product is unprofitable for the enterprise, since it is produced in amount of minimum bound of the production plan and the growth of its production volume will also lead to a decrease in total revenue of the enterprise by 2,64 monetary units per additional unit. Raw materials of the first and third types are scarce under this production plan and a further increase in their volumes per unit can lead to an increase in revenue by 3,1 and 2,7 monetary units. In this case, these volumes will vary between [157; 274,6] for raw materials of the first type and [68,75; 304,125] for raw materials of the third type.

6	Variable Cells						
7			Final	Reduced	Objective	Allowable	Allowable
8	Cell	Name	Value	Cost	Coefficient	Increase	Decrease
9	$B$10	Production 1	57,5	0	17	43	2,111111111
10	$C$10	Production 2	0	-1,35714286	16	1,357142857	1E+30
11	$D$10	Production 3	18	0	15	19	3,166666667
12	$E$10	Production 4	11	-2,64285714	12	2,642857143	1E+30
13							
14	Constraints						
15			Final	Shadow	Constraints	Allowable	Allowable
16	Cell	Name	Value	Price	R.H. Side	Increase	Decrease
17	$G$13	Raw materials 1	220	3,07142857	220	54,6	63
18	$G$14	Raw materials 2	230,5	0	250	1E+30	19,5
19	$G$15	Raw materials 3	270	2,71428571	270	34,125	201,25
20							

Fig. 4. Sensitivity report

Source: compiled by author's calculations

Thirdly, we construct the canonical form for the written model, where we forcefully add an inequality of an admissible solution for the second unprofitable product. Thus,

$2x_1 + 3x_2 + 4x_3 + 3x_4 + x_5 = 220$

$3x_1 + 2x_2 + 2x_3 + 2x_4 + x_6 = 250$

$4x_1 + 3x_2 + x_3 + 2x_4 + x_7 = 270$

$x_4 - x_8 + y_1 = 11$

$x_2 - x_9 + y_2 = 0$

$x_i \geq 0,\ i = \overline{1,9};\ y_1, y_2 \geq 0.$

$Z - 17x_1 - 16x_2 - 15x_3 - 12x_4 - M\,(y_1 + y_2) = 0.$

Fourthly, we write down the initial simplex table (Table 2).

Table 2

The initial simplex table of the resource utilization problem

Basic variables	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_5	2	3	4	3	1				
x_6	3	2	2	2	0	1			
x_7	4	3	1	2	0	0	1		
y_1	0	0	0	1	0	0	0	-1	0
y_2	0	1	0	0	0	0	0	0	-1
Z	–17	–16	–15	–12	0	0	0	0	0
–M	0	–M	0	–M	0	0	0	M	M

Source: compiled by author's calculations

The constructed basic of the initial simplex table contains five variables: x_5, x_6, x_7, y_1, y_2.

Fifthly, we determine the basic variables that will be in the optimal solution. The additional variables x_5, x_7, corresponding to the scarce resources RM_1 and RM_3, will be removed from the basic variables and will have values equal to zero. The variables x_8 and x_9, which demonstrate how more than the minimum output is produced, are also equal to zero and will not be included in the basic. The artificial variables y_1 and y_2 will be replaced by the corresponding variables that remain in the canonical form equations and will become the basic variables, i.e. $y_1 \leftrightarrow x_4$ and $y_2 \leftrightarrow x_2$. The variable x_6 will remain basic, as it corresponds to a non-scarce resource. And the basic variables will also include the two main variables that have non-zero values in the optimal solution. Thus, the basic variables will be as follows: x_1, x_2, x_3, x_4, x_6.

Sixthly, we build the matrix D from the vectors of the initial simplex table (Table 2) corresponding to the basic variables of the optimal solution (Table 3).

Table 3

The matrix *D*

	x_1	x_2	x_3	x_4	x_6
	2	3	4	3	0
	3	2	2	2	1
$D =$	4	3	1	2	0
	0	0	0	1	0
	0	1	0	0	0

Source: compiled by author's calculations

Accordingly, the inverse matrix will be presented in Table 4.

Table 4

The inverse matrix D^{-1}

	–1/14	0	2/7	–5/14	–9/14
	0	0	0	0	1
$D^{-1} =$	2/7	0	–1/7	–4/7	–3/7
	0	0	0	1	0
	–5/14	1	–4/7	3/14	11/14

Source: compiled by author's calculations

Multiplying the row vector (17; 16; 15; 12; 0), which consists of the coefficients of the objective function of the direct problem with the set of basic variables that we found in the optimal plan, we obtain the solution to the dual problem: Y* = (53/14; 0; 19/7; –37/14; –19/14) or convert to decimals (3,071; 0; 2,714; –2,643; –1,357), which completely coincides with the shadow prices given in sensitivity report (Fig. 4). The columns of a matrix correspond to additional variables x_5, x_6, x_7, x_8, x_9. Thus, the matrix is obtained, which can be used for a more detailed analysis of the impact of changes on the optimal solution of the linear planning problem.

It is necessary to ensure that the proposed algorithm allows us to find the correct set of basic variables. We present a simplex table (Table 5) which contains the optimal solution to the linear resource utilization problem for the developed canonical form. The optimality criterion for the simplex method assumes that the last row should contain non-negative numbers. The values in the columns correspond to the positive variables that are included in the canonical form with a minus sign and should have opposite signs.

Table 5

A simplex table of the optimal solution of a resource utilization problem

Basic variables	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_1	1	0	0	0	–1/14	0	2/7	5/14	9/14
x_2	0	1	0	0	0	0	0	0	–1
x_3	0	0	1	0	2/7	0	–1/7	4/7	3/7
x_4	0	0	0	1	0	0	0	–1	0
x_6	0	0	0	0	–5/14	1	–4/7	–3/14	–11/14
Z	0	0	0	0	53/14	0	19/7	37/14	19/14
Y	y_6	y_7	y_8	y_9	y_1	y_2	y_3	y_4	y_5

Source: compiled by author's calculations

This remark should be taken into account when conducting economic analysis of the solution of linear programming problems obtained by the simplex method, and implementing the proposed algorithm, this situation does not arise. In Table 5 is presented the solution to the dual problem. Given the above, we can conduct a more in-depth analysis of the obtained optimal solution to the linear programming problem of resource utilization.

According to the table 5, it can be concluded that an increase in the stocks of raw materials of the first type by one unit will cause an increase of revenue by 53/14 monetary units due to (column x_5) a decrease in the output of the first type by 1/14 units; an increase in the output of the third type by 2/7 units will lead to a decrease in the stocks of raw materials of the second type by 5/14 units. The production of an additional unit of unprofitable second and fourth types of products (for example, an additional unit of the fourth type of product (column x_8) will lead to a decrease in total sales revenue by 37/14 monetary units, while production of the first type of product will decrease by 5/14 units and the third type of product by 4/7 units and the unused raw material stocks of the second type will increase by 3/14 units.

Conclusions. The theory of duality is a powerful tool in the study of optimization problems. It provides a possibility to design the new algorithms for solving problems.

It should be noted that given algorithm can be applied only to non-degenerate dual pair of coupled linear programming problems that have a nonempty set of feasible solutions. The drawback of the algorithm is the implementation of artificial restrictions on the inalienability of unprofitable products to the canonical form, which leads to the identification of an additional dual variable and, accordingly, the determination of its value. However, such situation may arise if the constructed linear programming problem is solved by the simplex method with the calculation of all simplex tables. The proposed algorithm does not involve the use of the simplex method, and therefore this situation will not affect the economic analysis of the obtained solution. Despite the identified drawback, the proposed algorithm can be effectively used to conduct a more in-depth analysis of the solution of linear programming problems in order to determine the impact of changes in the given constraints on production in general.

The constructed inverse matrix can also be used to study the overall impact on the optimal solution when the right-hand sides of the constraint system are changed, a new type of product is included, or a new constraint is imposed on a new type of raw material, etc.

Thus, a specialist without any additional mathematical tools can conduct a more in-depth analysis of the production planning problem applying all the possibilities of the duality theory.

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